GROTHENDIECK CLASSES OF QUIVER VARIETIES

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Abstract. We prove a formula for the structure sheaf of a quiver variety in the Grothendieck ring of its embedding variety. This formula generalizes and gives new expressions for Grothendieck polynomials. We furthermore conjecture that the coefficients in our formula have signs which alternate with degree. The proof of our formula involves K-theoretic generalizations of several useful cohomological tools, including the Thom-Porteous formula, the Jacobi-Trudi formula, and a Gysin formula of Pragacz.

1. Introduction

Let X be a non-singular variety and $E_0 \to E_1 \to \cdots \to E_n$ a sequence of vector bundles and bundle maps over X. A set of *rank conditions* for this sequence is a collection $r = \{r_{ij}\}$ of non-negative integers for $0 \leq i < j \leq n$. This data defines a *quiver variety*:

$$\Omega_r = \Omega_r(E_\bullet) = \{x \in X \mid \operatorname{rank}(E_i(x) \to E_j(x)) \leq r_{ij} \ \forall i < j\} . \tag{1.1}$$

This set has a natural structure of subscheme of X. Namely, it is the scheme theoretic intersection of the zero sections of the bundle maps $\bigwedge^{r_{ij}+1} E_i \to \bigwedge^{r_{ij}+1} E_j$.

We will demand that the rank conditions can *occur*, i.e. there exists a sequence of vector spaces and linear maps $V_0 \to V_1 \to \cdots \to V_n$ such that $\dim V_i = \operatorname{rank} E_i$ and $\operatorname{rank}(V_i \to V_j) = r_{ij}$ for all $i < j$. If we set $r_{ii} = \operatorname{rank} E_i$, then this is equivalent to the conditions $r_{ij} \leq \min(r_{i,j-1}, r_{i+1,j})$ for $i < j$ and $r_{i+1,j-1} - r_{i,j-1} - r_{i+1,j} + r_{ij} \geq 0$ for $j - i \geq 2$.

The expected (and maximal possible) codimension of the quiver variety Ω_r is $d(r) = \sum_{i<j} (r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$. When this codimension is obtained, the main result of [5] gives a formula for the cohomology class of Ω_r:

$$[\Omega_r] = \sum_{|\mu| = d(r)} c_\mu(r)\, s_{\mu_1}(E_1 - E_0)\, s_{\mu_2}(E_2 - E_1) \cdots s_{\mu_n}(E_n - E_{n-1}) . \tag{1.2}$$

This sum is over sequences $\mu = (\mu_1, \dots, \mu_n)$ of n partitions such that the sum $|\mu| = \sum |\mu_i|$ of the weights of these partitions is equal to the expected codimension $d(r)$. (Recall that the weight of a partition is the sum of its part, or the number of boxes in its Young diagram.) If λ is a partition then $s_\lambda(E_i - E_{i-1})$ denotes the double Schur function $s_\lambda(x; y)$ applied to the Chern roots of the bundles E_i and E_{i-1}. The coefficients $c_\mu(r)$ are certain integers given by an explicit combinatorial algorithm. Surprisingly, these coefficients appear to be non-negative. It is conjectured

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in [5] that each coefficient counts the number of sequences of semistandard Young tableaux satisfying certain properties; this has been proved when the sequence $E_\bullet$ has at most four bundles [2].

While the cohomology class of a quiver variety Ω_r represents useful global information, there is more information hidden in its structure sheaf $\mathcal{O}_{\Omega_r}$. The best possible representation of this information that one could hope for might be an explicit resolution of the structure sheaf by locally free sheaves on X. Such a resolution would generalize fundamental constructions such as the Koszul complex and the Eagon-Northcott complex [6], at least up to quasi-isomorphism. Such resolutions, however, are known in only very few cases, such as for Schubert varieties in Grassmannians [17]. The main theorem in this paper is a formula for the structure sheaf $\mathcal{O}_{\Omega_r}$ of a quiver variety in the Grothendieck ring $K^\circ X$ of algebraic vector bundles on X. This corresponds to computing the alternating sum of a locally free resolution, so the K-theory formula contains the cohomology formula as its leading term.

Our formula has the form

$$[\mathcal{O}_{\Omega_r}] = \sum_{|\mu| \geq d(r)} c_\mu(r)\, G_{\mu_1}(E_1 - E_0) \cdots G_{\mu_n}(E_n - E_{n-1}) \in K^\circ X \tag{1.3}$$

where the sum is this time over a finite collection of sequences of partitions for which the weights add up to at least the expected codimension. The elements $G_{\mu_i}(E_i - E_{i-1}) \in K^\circ X$ are called stable Grothendieck polynomials; these will be defined in Section 2. The coefficients $c_\mu(r)$ in this formula are given by a generalization of the algorithm for the coefficients of (1.2). In particular, the coefficients are the same when $|\mu| = d(r)$.

We conjecture that the signs of the new coefficients alternate with the weight of μ, i.e. $(-1)^{|\mu|-d(r)} c_\mu(r) \geq 0$. It appears to be a rather general phenomenon that coefficients which show up in K-theoretic formulas tend to have alternating signs, although this is very poorly understood. For example, a formula of Fomin and Kirillov shows that the signs of the coefficients in Grothendieck polynomials alternate with degree [9]. Similarly we have proved in [3] that the structure constants of the Grothendieck ring of a Grassmann variety with respect to its basis of Schubert structure sheaves have signs which alternate with codimension. In fact, this is a special case of our conjecture, since said structure constants are special cases of the coefficients $c_\mu(r)$ of (1.3). It is worth pointing out that in all cases where alternation of signs in K-theory has been proved, this has been achieved by giving explicit formulas for the coefficients in question. This is in contrast to cohomology, where positivity results can often be obtained by realizing coefficients as the number of points in an intersection of varieties in general position.

Our conjecture is true when Ω_r is a variety of complexes, i.e. $r_{ij} = 0$ whenever $j - i \geq 2$. In fact, the algorithm for the coefficients $c_\mu(r)$ is particularly simple in this case, and it shows that (1.3) is multiplicity free in the sense that every coefficient $c_\mu(r)$ is either 1, -1, or zero. We have furthermore verified the conjecture computationally for all sequences with at most 4 bundles of ranks up to 7.

The proof of the cohomology formula in [5] is based on the simple idea of realizing the quiver variety Ω_r as a birational image of a simpler quiver variety $\Omega_{\tilde{r}}$ which lives on a product of Grassmann bundles over X. The class of Ω_r can then be calculated inductively as the pushforward of the class of $\Omega_{\tilde{r}}$, which is done using a Gysin formula of Pragacz [21]. However, before this Gysin formula can be applied, one

must first rearrange the inductive formula for $\Omega_{\tilde{r}}$ by replacing the Schur polynomials $s_{\mu_i}(E_i - E_{i-1})$ in this formula with linear combinations of products $s_\sigma(E_i - F) \cdot s_\tau(F - E_{i-1})$ for other bundles F, which can be done by invoking the coproduct in the ring of symmetric functions. Thus the cohomology formula is a consequence of the large cohomological toolbox surrounding the ring of symmetric functions, once the right overall geometric construction has been made. Of particular importance are the coproduct on Schur functions and Pragacz's Gysin formula, as well as the Thom-Porteous formula for starting the induction.

While the same method turns out to work for the K-theory formula, it was far from obvious that this would be possible when we started our project. First of all, while double stable Grothendieck polynomials had been defined by Fomin and Kirillov [9, 8] and studied combinatorially [7], they had never been applied to geometry. Furthermore, the properties of Schur functions that are needed for the cohomology formula had no known analogues. Our work on generalizing the formula has therefore consisted mainly of finding and proving K-theoretic analogues of known cohomological tools.

The first step in this direction was carried out in [3] where we proved that the linear span of all stable Grothendieck polynomials form a bialgebra Γ which is a K-theory parallel of the ring of symmetric functions. In the same way as the ring of symmetric functions describes cohomology of Grassmannians, Γ describes their K-theory. In this paper we prove a K-theory version of the Thom-Porteous formula, a Gysin formula for calculating K-theoretic pushforwards from a Grassmann bundle which generalizes Pragacz's cohomological formula, and we develop the few extra bits of combinatorics which make it all fit together.

One additional ingredient in the proofs of (1.2) and (1.3) is a result of Lakshmibai and Magyar showing that a quiver variety of the expected codimension is Cohen-Macaulay [16]. For the K-theory formula we shall furthermore need their result about rational singularities of quiver varieties to deduce that the structure sheaf of the inductive quiver variety $\Omega_{\tilde{r}}$ mentioned above pushes forward to the structure sheaf of Ω_r.

In Section 2 we fix the notation regarding Grothendieck polynomials and stable Grothendieck polynomials, and we explain their relations to geometry. In Section 3 we define stable Grothendieck polynomials for arbitrary sequences of integers which extend the definition of stable Grothendieck polynomials for partitions. This is needed for describing the algorithm for the coefficients in (1.3). This algorithm is then presented in Section 4, where we also explain the meaning of our formula when X is singular or Ω_r does not have its expected codimension. In addition we interpret the formula in the case of varieties of complexes. In Section 5 we show that Grothendieck polynomials and stable Grothendieck polynomials are special cases of the quiver formula. Combined with some recent results of Lascoux [18], this supplies additional evidence for our conjecture about the signs of the coefficients $c_\mu(r)$. The last two sections are devoted to proving our generalization of Pragacz's Gysin formula. Section 6 proves a generalization of the Jacobi-Trudi formula for Schur functions, which in Section 7 is used to establish the Gysin formula itself. We finish the paper by noticing that the pushforward map from a Grassmann bundle is multiplicative when applied to products of Grothendieck polynomials for short partitions.

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polynomials. In addition we thank W. Fulton for numerous helpful comments, suggestions, and encouragements during the project.

2. Grothendieck polynomials

In this section we fix the notation concerning Grothendieck polynomials and stable Grothendieck polynomials, and explain their relations to geometry. We furthermore summarize the necessary results from [3].

Given a permutation $w \in S_n$, Lascoux and Schützenberger define the *double Grothendieck polynomial* $\mathfrak{G}_w = \mathfrak{G}_w(x;y)$ for w as follows [19]. For the longest permutation $w_0 = n\,(n-1)\cdots 2\,1$ we set

$$\mathfrak{G}_{w_0} = \prod_{i+j\leq n} (x_i + y_j - x_i y_j)\,.$$

If w is not the longest permutation, we can find a simple reflection $s_i = (i, i{+}1) \in S_n$ such that $\ell(ws_i) = \ell(w){+}1$. Here $\ell(w)$ denotes the length of w, which is the smallest number ℓ for which w can be written as a product of ℓ simple reflections. We then define

$$\mathfrak{G}_w = \pi_i(\mathfrak{G}_{ws_i})$$

where π_i is the isobaric divided difference operator given by

$$\pi_i(f) = \frac{(1-x_{i+1}) f(x_1, x_2, \dots) - (1-x_i) f(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}\,.$$

This definition is independent of our choice of simple reflection s_i since the operators π_i satisfy the Coxeter relations.

Notice that the longest element in S_{n+1} is $w_0^{(n+1)} = w_0 \cdot s_n \cdot s_{n-1} \cdots s_1$. Since $\pi_n \cdot \pi_{n-1} \cdots \pi_1$ applied to the Grothendieck polynomial for $w_0^{(n+1)}$ is equal to $\mathfrak{G}_{w_0}$, it follows that $\mathfrak{G}_w$ does not depend on which symmetric group w is considered an element of.

Now let $F_1 \subset F_2 \subset \cdots \subset F_n \xrightarrow{\varphi} H_n \twoheadrightarrow \cdots \twoheadrightarrow H_2 \twoheadrightarrow H_1$ be a full flag of vector bundles on X followed by a map φ to a dual full flag. For $w \in S_{n+1}$ we define the degeneracy locus

$$\Omega_w = \Omega_w(F_\bullet \to H_\bullet) = \{x \in X \mid \operatorname{rank}(F_q(x) \to H_p(x)) \leq r_w(p,q)\ \forall p, q\}$$

where $r_w(p,q) = \#\{i \leq p \mid w(i) \leq q\}$. The expected codimension for this locus is the length of w.

Suppose $F_1 \subset F_2 \subset \cdots \subset F_n \subset V$ is a full flag of subbundles in a vector bundle V of rank $n+1$. Let $\pi : \mathrm{F}\ell^*(V) \to X$ be the bundle of dual flags of V, with tautological flag $\pi^* V \twoheadrightarrow \tilde{H}_n \twoheadrightarrow \cdots \twoheadrightarrow \tilde{H}_1$. In this case the Schubert variety $\tilde{\Omega}_w = \Omega_w(\pi^* F_\bullet \to \tilde{H}_\bullet)$ has codimension $\ell(w)$ in $\mathrm{F}\ell^*(V)$. Fulton and Lascoux [11] have proved that its structure sheaf is given by the double Grothendieck polynomial for w:

$$[\mathcal{O}_{\tilde{\Omega}_w}] = \mathfrak{G}_w(1 - \tilde{L}_1^{-1}, \dots, 1 - \tilde{L}_n^{-1}; 1 - M_1, \dots, 1 - M_n) \tag{2.1}$$

in $K^\circ\,\mathrm{F}\ell^*(V)$, where $\tilde{L}_i = \ker(\tilde{H}_i \to \tilde{H}_{i-1})$ and $M_i = F_i / F_{i-1}$.

Using the fact that a Grothendieck polynomial $\mathfrak{G}_w(x;y)$ does not depend on which symmetric group w belongs to, this formula readily generalizes as follows:

Theorem 2.1. *If the locus* $\Omega_w = \Omega_w(F_\bullet \to H_\bullet)$ *in* X *has its expected codimension* $\ell(w)$, *then*

$$[\mathcal{O}_{\Omega_w}] = \mathfrak{G}_w(1-L_1^{-1},\dots,1-L_n^{-1};1-M_1,\dots,1-M_n)$$

where $L_i = \ker(H_i \to H_{i-1})$ *and* $M_i = F_i/F_{i-1}$.

Proof. Set $V = F_n \oplus H_n$ and let $\pi : \mathrm{F}\ell^*(V) \to X$ be the dual flag bundle of V with tautological flag $\pi^* V \twoheadrightarrow \tilde{H}_{2n-1} \twoheadrightarrow \cdots \twoheadrightarrow \tilde{H}_1$. Define $\psi : F_n \to V$ by $\psi(\sigma) = (\sigma, \varphi(\sigma))$, and set $F_i = \psi(F_n) + \ker(H_n \to H_{2n-i})$ for $n < i < 2n$. Then $F_1 \subset \cdots \subset F_{2n-1} \subset V$ is a full flag of subbundles in V, and (2.1) applies to give a formula for the structure sheaf of the Schubert variety $\tilde{\Omega}_w = \Omega_w(\pi^* F_\bullet \to \tilde{H}_\bullet)$.

Set $H_i = (F_n/F_{2n-i}) \oplus H_n$ for $n < i < 2n$. Then there is a unique section $s : X \to \mathrm{F}\ell^*(V)$ such that the dual flag $V \twoheadrightarrow H_{2n-1} \twoheadrightarrow \cdots \twoheadrightarrow H_1$ is the pullback of the tautological flag $\pi^* V \twoheadrightarrow \tilde{H}_\bullet$ on $\mathrm{F}\ell^*(V)$, and furthermore we have $\Omega_w = s^{-1}(\tilde{\Omega}_w)$ as subschemes of X. Since the loci Ω_w and $\tilde{\Omega}_w$ have the same codimensions and are Cohen-Macaulay, this implies that

$$\begin{aligned}
[\mathcal{O}_{\Omega_w}] &= s^*[\mathcal{O}_{\tilde{\Omega}_w}] = s^*\,\mathfrak{G}_w(1-\tilde{L}_1^{-1},\dots,1-\tilde{L}_{2n-1}^{-1};1-M_1,\dots,1-M_{2n-1}) \\
&= s^*\,\mathfrak{G}_w(1-\tilde{L}_1^{-1},\dots,1-\tilde{L}_n^{-1};1-M_1,\dots,1-M_n) \\
&= \mathfrak{G}_w(1-L_1^{-1},\dots,1-L_n^{-1};1-M_1,\dots,1-M_n)
\end{aligned}$$

which completes the proof. □

We now turn to stable Grothendieck polynomials. Given a permutation $w \in S_n$ and a non-negative integer m, we let $1^m \times w \in S_{m+n}$ denote the shifted permutation which is the identity on $\{1,2,\dots,m\}$ and which maps j to $w(j-m)+m$ for $j > m$. Fomin and Kirillov have shown that when m grows to infinity, the coefficient of each fixed monomial in $\mathfrak{G}_{1^m \times w}$ eventually becomes stable [9]. The *double stable Grothendieck polynomial* $G_w \in \mathbb{Z}[\![x_i, y_i]\!]_{i \geq 1}$ is defined as the resulting power series:

$$G_w = G_w(x;y) = \lim_{m\to\infty} \mathfrak{G}_{1^m \times w}\,.$$

Fomin and Kirillov also proved that this power series is symmetric in the variables $\{x_i\}$ and $\{y_i\}$ separately, and that

$$G_w(1-e^{-x};1-e^{y}) = G_w(1-e^{-x_1},1-e^{-x_2},\dots;1-e^{y_1},1-e^{y_2},\dots)$$

is super symmetric, i.e. if one sets $x_1 = y_1$ in this expression then the result is independent of x_1 and y_1. Alternatively, these facts can be deduced from Theorem 2.1.

We shall be mostly concerned with stable Grothendieck polynomials for Grassmannian permutations. If λ is a partition and $p \geq \ell(\lambda)$, i.e. $\lambda_{p+1} = 0$, the *Grassmannian permutation* for λ with descent in position p is the unique permutation w_λ such that $w_\lambda(i) = i + \lambda_{p+1-i}$ for $1 \leq i \leq p$ and $w_\lambda(i) < w_\lambda(i+1)$ for $i \neq p$. We define $G_\lambda = G_{w_\lambda}$. Notice that if $q > p$, then the Grassmannian permutation for λ with descent at position q is equal to $1^{q-p} \times w_\lambda$. Therefore G_λ is independent of the choice of p.

Let $\Gamma \subset \mathbb{Z}[\![x_i, y_i]\!]$ be the linear span of all stable Grothendieck polynomials. It is shown in [3] that this group is a bialgebra and that the elements G_λ form a basis. We will proceed to describe the structure constants of Γ.

If a and b are two non-empty subsets of the positive integers $\mathbb{N}$, we will write $a < b$ if $\max(a) < \min(b)$, and $a \leq b$ if $\max(a) \leq \min(b)$. We define a *set-valued tableau* to be a labeling of the boxes in a Young diagram or skew diagram with finite non-empty subsets of $\mathbb{N}$, such that the rows are weakly increasing from left

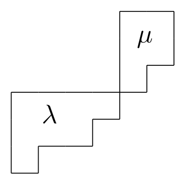
to right and the columns strictly increasing from top to bottom. The shape $\mathrm{sh}(T)$ of a tableau T is the partition or skew diagram it is a labeling of. For example,

$$\begin{array}{|c|c|c|c|}\hline \multicolumn{2}{c|}{} & 1 & 23 \\ \hline \multicolumn{1}{c|}{} & 12 & 234 \\ \cline{1-3} 2 & 35 & 7 \\ \cline{1-3}\end{array}$$

is a set-valued tableau whose shape is the skew diagram between the partitions $(4, 3, 3)$ and $(2, 1)$. The *word* of a set-valued tableau is the sequence of integers in its boxes when these are read left to right and then bottom to top, and the integers in a single box are arranged in increasing order. The word of the above tableau is $(2, 3, 5, 7, 1, 2, 2, 3, 4, 1, 2, 3)$.

We say that a sequence of positive integers $w = (i_1, i_2, \dots, i_\ell)$ is has *content* $(c_1, c_2, \dots, c_r)$ if w consists of c_1 1's, c_2 2's, and so on up to c_r r's. If the content of each subsequence $(i_k, \dots, i_\ell)$ of w is a partition, then w is called a *reverse lattice word*.

If λ and μ are partitions, we let $\lambda * \mu$ denote the skew diagram obtained by attaching the Young diagrams for λ and μ corner to corner as shown:

$\lambda * \mu \;=$ 

The main result of [3] now says that

$$G_\lambda \cdot G_\mu = \sum_\nu c^\nu_{\lambda\mu}\, G_\nu \tag{2.2}$$

where $c^\nu_{\lambda\mu}$ is equal to $(-1)^{|\nu|-|\lambda|-|\mu|}$ times the number of set-valued tableaux T of shape $\lambda * \mu$ such that the word of T is a reverse lattice word with content ν.

Now if λ, μ, and ν are partitions, we set $d^\nu_{\lambda\mu} = c^\rho_{\nu R}$, where $R = (p)^q$ is any rectangular partitions containing λ and μ, and $\rho = (p+\lambda_1, \dots, p+\lambda_q, \mu_1, \mu_2, \dots)$ is the partition obtained by attaching λ and μ to the sides of R. In [3] it is proved that these coefficients do not depend on the choice of the rectangle R. Furthermore, whenever x, y, z, and w are different sets of variables we have

$$G_\nu(x, y; z, w) = \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, G_\lambda(x; z) \cdot G_\mu(y; w)\,. \tag{2.3}$$

Theorem 2.2. *The group $\Gamma = \bigoplus_\lambda G_\lambda \subset \mathbb{Z}[\![x_i, y_i]\!]$ is a commutative and cocommutative bialgebra with unit and counit. Multiplication is given by (2.2) and the coproduct $\Delta : \Gamma \to \Gamma \otimes \Gamma$ is defined by $\Delta G_\nu = \sum_{\lambda,\mu} d^\nu_{\lambda\mu} G_\lambda \otimes G_\mu$.*

It is also possible to give a formula for stable Grothendieck polynomials based on set-valued tableaux. Given a tableau T, let x^T be the monomial in which the exponent of x_i is the number of boxes in T which contain the integer i. If T is the tableau displayed above we get $x^T = x_1^2\, x_2^4\, x_3^3\, x_4\, x_5\, x_7$. We let $|T|$ denote the total degree of this monomial, i.e. the sum of the cardinalities of the sets in the boxes of T. In [3] it is proved that the single stable Grothendieck polynomial $G_\lambda(x) = G_\lambda(x; 0)$

is given by

$$G_\lambda(x) = \sum_{\mathrm{sh}(T)=\lambda} (-1)^{|T|-|\lambda|}\, x^T \tag{2.4}$$

where the sum is over all set-valued tableaux T of shape λ. The double stable Grothendieck polynomial for λ is then given by

$$G_\lambda(x;y) = \sum_{\sigma,\tau} d^\lambda_{\sigma\tau}\, G_\sigma(x)\cdot G_{\tau'}(y)$$

where τ' denotes the conjugate partition of τ.

Let $F = L_1 \oplus \cdots \oplus L_f$ and $E = M_1 \oplus \cdots \oplus M_e$ be vector bundles on a variety X which are both direct sums of line bundles. We then define

$$G_\lambda(F-E) = G_\lambda(1-L_1^{-1},\dots,1-L_f^{-1};1-M_1,\dots,1-M_e) \ \in\ K^\circ(X)\,.$$

Since G_λ is symmetric, this is a polynomial in the exterior powers of $F^\vee$ and E. Therefore the definition makes sense also when E and F are not direct sums of line bundles. For example we have $G_1(F-E) = 1 - \frac{\bigwedge^e E}{\bigwedge^f F}$. The fact that $G_\lambda(1-e^{-x};1-e^y)$ is super-symmetric implies that $G_\lambda(F\oplus H - E\oplus H) = G_\lambda(F-E)$ for any bundle H. In particular we can regard G_λ as a well defined function $G_\lambda : K^\circ X \to K^\circ X$. Equation (2.3) then says that for any elements $\alpha,\beta \in K^\circ X$ we have

$$G_\lambda(\alpha+\beta) = \sum_{\sigma,\tau} d^\lambda_{\sigma\tau}\, G_\sigma(\alpha)\cdot G_\tau(\beta)\,. \tag{2.5}$$

Another useful fact, due to Fomin, is that $G_\lambda(F-E) = G_{\lambda'}(E^\vee - F^\vee)$.

This notation makes it possible to give a Thom-Porteous formula for K-theory which is analogous to its cohomological equivalent. Let $E \to F$ be a morphism between vector bundles of ranks e and f. Given an integer $r \le \min(e,f)$ we have the degeneracy locus $\Omega_r(E\to F) = \{x \in X \mid \mathrm{rank}(E(x)\to F(x)) \le r\}$.

Theorem 2.3. *If the codimension of $\Omega_r(E\to F)$ in X is $(e-r)(f-r)$ then the class of its structure sheaf is given by*

$$[\mathcal{O}_{\Omega_r(E\to F)}] = G_\lambda(F-E)$$

where $\lambda = (e-r)^{f-r}$ is a rectangular partition with $f-r$ rows and $e-r$ columns.

Proof. By the splitting principle we may assume that E and F come equipped with full flags $E_1 \subset \cdots \subset E_e = E$ and $F = F_f \twoheadrightarrow \cdots \twoheadrightarrow F_1$. Let w_λ be the Grassmannian permutation for λ with descent at position f. Then w is a permutation in S_n where $n = e+f-r$. Set $E_i = E \oplus \mathcal{O}_X^{\oplus i-e}$ for $e < i < n$ and $F_j = F \oplus \mathcal{O}_X^{\oplus j-f}$ for $f < j < n$, and let $\varphi : E_{n-1} \to F_{n-1}$ be the map $E_{n-1} \to E \to F \to F_{n-1}$, i.e. the map $E \to F$ is extended by zeros on the trivial parts of E_{n-1} and F_{n-1}. It is now easy to check that $\Omega_r(E\to F) = \Omega_{w_\lambda}(E_\bullet \to F_\bullet)$ as subschemes of X, so by Theorem 2.1 we get

$$[\mathcal{O}_{\Omega_r(E\to F)}] = \mathfrak{G}_{w_\lambda}(1-L_1^{-1},\dots,1-L_f^{-1},0,\dots,0;1-M_1,\dots,1-M_e,0,\dots,0)$$

where $L_j = \ker(F_j \to F_{j-1})$ and $M_i = E_i/E_{i-1}$. Notice that $\Omega_{w_\lambda}(E_\bullet \to F_\bullet) = \Omega_{1\times w_\lambda}(E_\bullet \subset E_{n-1}\oplus \mathcal{O}_X \xrightarrow{\varphi\oplus 1} F_{n-1}\oplus\mathcal{O}_X \twoheadrightarrow F_\bullet)$. This means that the formula does not change when we shift the permutation w_λ, so in fact we have

$$[\mathcal{O}_{\Omega_r(E\to F)}] = G_{w_\lambda}(1-L_1^{-1},\dots,1-L_f^{-1};1-M_1,\dots,1-M_e) = G_\lambda(F-E)\,.$$

This finishes the proof. $\square$

3. Sequences of integers

In order to define the coefficients in our formula for quiver varieties, we need to define stable Grothendieck polynomials for arbitrary sequences of integers. Our definition of these is inspired by the following determinant formula of Lenart.

For integers $k \in \mathbb{Z}$ and $i \geq 0$, let $h_k(x_1, \dots, x_n/1^i)$ denote the coefficient of t^k in the formal power series expansion of

$$\frac{(1-t)^i}{\prod_{j=1}^n (1 - x_j t)} \,.$$

In particular we have $h_0(x_1, \dots, x_n/1^i) = 1$ and $h_k(x_1, \dots, x_n/1^i) = 0$ for $k < 0$.

Let $I = (I_1, I_2, \dots, I_\ell)$ be a finite sequence of integers of length ℓ. For convenience we shall regard I_j as being zero if $j > \ell$. For $n > \ell$ we now define $\mathcal{G}_I(x_1, \dots, x_n)$ to be the determinant of the $n \times n$ matrix whose (i,j)'th entry is equal to $h_{I_i+j-1}(x_1, \dots, x_n/1^{i-1})$:

$$\mathcal{G}_I(x_1, \dots, x_n) = \det \left(h_{I_i+j-1}(x_1, \dots, x_n/1^{i-1}) \right)_{1 \leq i,j \leq n}$$

Notice that the size of this determinant depends on the number of variables. With this notation we have:

Theorem 3.1 (Lenart [20]). *If I is a partition then*

$$\mathcal{G}_I(x_1, \dots, x_n) = \mathfrak{G}_{w_I}(x) = G_I(x_1, \dots, x_n) \,.$$

Lemma 3.2. *Let I and J be sequences of integers and suppose $p < q$ are integers. Then*

$$\mathcal{G}_{I,p,q,J}(x_1, \dots, x_n) = \sum_{k=p+1}^{q} \mathcal{G}_{I,q,k,J}(x_1, \dots, x_n) - \sum_{k=p+1}^{q-1} \mathcal{G}_{I,q-1,k,J}(x_1, \dots, x_n) \,.$$

Proof. To cut down on the notation, we shall prove this in the case where I and J are empty and $n = 2$. The proof of the general case is exactly the same. For convenience we will also write $h_k(x/1^i)$ for $h_k(x_1, x_2/1^i)$.

Using the rule $h_k(x) = h_{k+1}(x) - h_{k+1}(x/1)$ repeatedly we get

$$\begin{vmatrix} h_p(x) & h_{p+1}(x) \\ h_q(x) & h_{q+1}(x) \end{vmatrix} = \sum_{k=p+1}^{q} \begin{vmatrix} -h_k(x/1) & -h_{k+1}(x/1) \\ h_q(x) & h_{q+1}(x) \end{vmatrix} = \sum_{k=p+1}^{q} \mathcal{G}_{q,k}(x_1, x_2) \,.$$

The lemma follows from this since

$$\begin{vmatrix} h_p(x) & h_{p+1}(x) \\ h_q(x/1) & h_{q+1}(x/1) \end{vmatrix} = \begin{vmatrix} h_p(x) & h_{p+1}(x) \\ h_q(x) & h_{q+1}(x) \end{vmatrix} - \begin{vmatrix} h_p(x) & h_{p+1}(x) \\ h_{q-1}(x) & h_q(x) \end{vmatrix} \,.$$

□

Corollary 3.3. *Let $I = (I_1, I_2, \dots, I_\ell)$ be a sequence of integers and let n be an integer such that $n \geq \ell$ and $n \geq i - I_i$ for all $1 \leq i \leq \ell$. Then $\mathcal{G}_I(x_1, \dots, x_n)$ is a finite linear combination of determinants $\mathcal{G}_\lambda(x_1, \dots, x_n)$ for partitions λ:*

$$\mathcal{G}_I(x_1, \dots, x_n) = \sum_\lambda \delta_{I,\lambda} \, \mathcal{G}_\lambda(x_1, \dots, x_n) \,.$$

Furthermore the coefficients $\delta_{I,\lambda}$ do not depend on n.

Proof. Define a "potential" function $\rho(I) = \sum_{j=1}^n (n-j)(\bar{I}_j - I_j)$ where $\bar{I}_j = \max\{I_k \mid j \leq k \leq n\}$. Then $\rho(I) \geq 0$ for all sequences I.

We proceed by induction on $\rho(I)$. If $\rho(I) = 0$ then I must be weakly decreasing. In fact it must be a partition because the assumption $n \geq n - I_n$ implies that $I_n \geq 0$. Therefore $\mathcal{G}_I(x_1, \dots, x_n)$ already has the desired form.

If $\rho(I) > 0$ then for some $1 \leq j < n$ we must have $I_j < I_{j+1}$. We can now apply Lemma 3.2 with $p = I_j$ and $q = I_{j+1}$ to write $\mathcal{G}_I(x_1, \dots, x_n)$ as a linear combination of other determinants $\mathcal{G}_J(x_1, \dots, x_n)$, and it is easy to check that these satisfy $\rho(J) < \rho(I)$ and $n \geq i - J_i$ for all $1 \leq i \leq \ell$. Each of these new determinants is therefore a linear combination of the polynomials $\mathcal{G}_\lambda(x_1, \dots, x_n)$ by induction, which proves the claim for the sequence I.

The fact that the coefficients $\delta_{I,\lambda}$ are independent of n follows because the formula of Lemma 3.2 is independent of n. □

Now define $G_I = G_I(x;y) = \sum_\lambda \delta_{I,\lambda}\, G_\lambda(x;y) \in \Gamma$. This is well defined by the corollary, and since $G_I(x_1, \dots, x_n) = \mathcal{G}_I(x_1, \dots, x_n)$ when n is sufficiently large, we have $G_I(x) = \lim_{n\to\infty} \mathcal{G}_I(x_1, \dots, x_n)$. Furthermore, Lemma 3.2 implies that

$$G_{I,p,q,J} = \sum_{k=p+1}^{q} G_{I,q,k,J} - \sum_{k=p+1}^{q-1} G_{I,q-1,k,J} \tag{3.1}$$

whenever $p < q$. This gives a practical way to compute the polynomials G_I.

Example 3.4. $G_{1\,1\,3} = G_{1\,3\,2} + G_{1\,3\,3} - G_{1\,2\,2} = G_{3\,2\,2} + G_{3\,3\,2} + G_{3\,3\,3} - 2\,G_{2\,2\,2}$.

Contrary to the case of Schur functions, a stable Grothendieck polynomial G_I for a sequence of integers is never equal to zero. In fact, (3.1) readily implies that $\sum_\lambda \delta_{I,\lambda} = 1$ for any sequence of integers I. It is also easy to prove that if J is a sequence of non-positive integers, then $G_{I,J} = G_I$. In addition, if G_λ occurs in the expansion of G_I, then λ must be contained in the partition $\bar{I} = (\bar{I}_1, \bar{I}_2, \dots)$, and furthermore we have $\delta_{I,\bar{I}} = 1$. A lower bound on λ may also be obtained. Let $\rho = (0, 1, 2, \dots)$ and let J denote the sequence $I - \rho = (I_1, I_2 - 1, I_3 - 2, \dots)$ arranged in decreasing order. Then any partition λ for which G_λ occurs in G_I must contain the partition $\tilde{I} = \overline{J + \rho}$. This lower bound is not sharp. If we take $I = (0, 2, 0, 3)$ then $\tilde{I} = (2, 2, 2, 1)$ and $\delta_{I,\tilde{I}} = 0$. We will not need these remarks in the following.

4. A formula for quiver varieties

We are now ready to describe our formula for the structure sheaf of a quiver variety. Let X be any Noetherian scheme equipped with a sequence $E_\bullet$ of vector bundles, and let $\Omega_r = \Omega_r(E_\bullet)$ be the associated quiver variety. We define a localized class $\boldsymbol{\Omega}_r$ in the Grothendieck group $K_\circ \Omega_r$ of coherent sheaves on Ω_r as follows. On the bundle $H = \operatorname{Hom}(E_0, E_1) \times_X \cdots \times_X \operatorname{Hom}(E_{n-1}, E_n) \xrightarrow{\pi} X$ we have a sequence of tautological maps $\pi^* E_0 \to \pi^* E_1 \to \cdots \to \pi^* E_n$. We let $\tilde{\Omega}_r \subset H$ denote the quiver variety defined by this sequence. Now the bundle maps on X define a section $s : X \to H$, and $\Omega_r = s^{-1}(\tilde{\Omega}_r)$. The localized class $\boldsymbol{\Omega}_r$ is defined by

$$\boldsymbol{\Omega}_r = s^!([\mathcal{O}_{\tilde{\Omega}_r}]) = \sum_{j\geq 0} (-1)^j \operatorname{Tor}_j^H(\mathcal{O}_X, \mathcal{O}_{\tilde{\Omega}_r}) \in K_\circ \Omega_r\,. \tag{4.1}$$

Notice that since s is a regular embedding, it follows that locally on H the structure sheaf of X has a finite free resolution, so the sum in (4.1) is finite. The definition

of $\mathbf{\Omega}_r$ implies that these classes are compatible with perfect pullback and proper pushforward [12].

The codimension of $\tilde{\Omega}_r$ in H is always equal to $d(r)$ [5]. Furthermore, Lakshmibai and Magyar have shown that this locus is Cohen-Macaulay and has rational singularities if X has these properties [16]. If X is Cohen-Macaulay and Ω_r has its expected codimension $d(r)$ in X, this implies that Ω_r is Cohen-Macaulay as well. In addition, a local regular sequence generating the ideal of X in H pulls back to a local regular sequence defining the ideal of Ω_r in $\tilde{\Omega}_r$ [10, Lemma A.7.1]. It follows from this that $\operatorname{Tor}_j^H(\mathcal{O}_X, \mathcal{O}_{\tilde{\Omega}_r}) = 0$ for all $j > 0$, so $\mathbf{\Omega}_r = [\mathcal{O}_X \otimes_{\mathcal{O}_H} \mathcal{O}_{\tilde{\Omega}_r}] = [\mathcal{O}_{\Omega_r}]$. More generally, this is true without the Cohen-Macaulay condition if $\operatorname{depth}(\Omega_r, X) = d(r)$ [10, Ex. 14.3.1]. We can now state our main result.

Theorem 4.1. *The image of $\mathbf{\Omega}_r$ in $K_\circ X$ is given by*

$$\mathbf{\Omega}_r = \sum_{|\mu| \geq d(r)} c_\mu(r)\, G_{\mu_1}(E_1 - E_0) \cdots G_{\mu_n}(E_n - E_{n-1})\,.$$

The sum is over a finite number of sequences μ of partitions μ_i such that the sum of the weights of these partitions is at least equal to $d(r)$. The coefficients $c_\mu(r)$ are integers which are given by an explicit combinatorial algorithm.

The algorithm which computes the coefficients $c_\mu(r)$ is the same as the one computing the coefficients in the cohomology formula [5], except the bialgebra Γ replaces the ring of symmetric functions. To describe the algorithm, we will construct an element P_r in the nth tensor power of Γ, such that

$$P_r = \sum_\mu c_\mu(r)\, G_{\mu_1} \otimes \cdots \otimes G_{\mu_n}\,.$$

It is convenient to arrange the rank conditions in a *rank diagram*:

$$\begin{array}{ccccccccc}
E_0 & \to & E_1 & \to & E_2 & \to & \cdots & \to & E_n \\
r_{00} & & r_{11} & & r_{22} & & \cdots & & r_{nn} \\
& r_{01} & & r_{12} & & \cdots & & r_{n-1,n} & \\
& & r_{02} & & \cdots & & r_{n-2,n} & & \\
& & & \ddots & & & & & \\
& & & & r_{0n} & & & &
\end{array}$$

In this diagram we replace each small triangle of numbers

$$\begin{array}{ccc}
r_{i,j-1} & & r_{i+1,j} \\
& r_{ij} &
\end{array}$$

with a rectangle R_{ij} with $r_{i+1,j} - r_{ij}$ rows and $r_{i,j-1} - r_{ij}$ columns.

$$R_{ij} = \underbrace{\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \end{array}}_{r_{i,j-1} - r_{ij}}\; r_{i+1,j} - r_{ij}$$

These rectangles are then arranged in a *rectangle diagram*:

$$\begin{array}{ccccccc}
R_{01} & & R_{12} & & \cdots & & R_{n-1,n} \\
& R_{02} & & \cdots & & R_{n-2,n} & \\
& & \ddots & & & & \\
& & & R_{0n} & & &
\end{array}$$

The combinatorial data contained in the rank conditions $r = \{r_{ij}\}$ is very well represented by this diagram. First of all, the rank conditions can occur if and only if the rectangles always get shorter when one travels south-east, while they get narrower when one travels south-west. Furthermore, the expected codimension $d(r)$ is equal to the sum of the areas of the rectangles R_{ij}. Finally, the element P_r depends only on the rectangle diagram.

We will define $P_r \in \Gamma^{\otimes n}$ by induction on n. When $n = 1$ (corresponding to a sequence of two vector bundles), the rectangle diagram has only one rectangle $R = R_{01}$. In this case we set

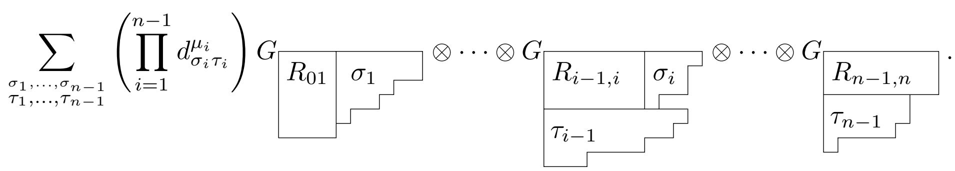
$$P_r = G_R \in \Gamma^{\otimes 1}$$

where R is identified with the partition for which it is the Young diagram. This case recovers the Thom-Porteous formula (Theorem 2.3).

If $n \geq 2$ we let $\bar{r}$ denote the bottom n rows of the rank diagram. Then $\bar{r}$ is a valid set of rank conditions, so by induction we can assume that

$$P_{\bar{r}} = \sum_{\mu} c_\mu(\bar{r})\, G_{\mu_1} \otimes \cdots \otimes G_{\mu_{n-1}} \tag{4.2}$$

is a well defined element of $\Gamma^{\otimes n-1}$. Now P_r is obtained from $P_{\bar{r}}$ by replacing each basis element $G_{\mu_1} \otimes \cdots \otimes G_{\mu_{n-1}}$ in (4.2) with the sum



This sum is over all partitions $\sigma_1, \dots, \sigma_{n-1}$ and $\tau_1, \dots, \tau_{n-1}$ such that σ_i has fewer rows than $R_{i-1,i}$ and the coproduct structure constant $d^{\mu_i}_{\sigma_i\tau_i}$ of Γ is non-zero. A diagram consisting of a rectangle $R_{i-1,i}$ with (the Young diagram of) a partition σ_i attached to its right side, and τ_{i-1} attached beneath should be interpreted as the sequence of integers giving the number of boxes in each row of this diagram.

It can happen that the rectangle $R_{i-1,i}$ is empty, since the number of rows or columns can be zero. If the number of rows is zero, then σ_i is required to be empty, and the diagram is the Young diagram of τ_{i-1}. If the number of columns is zero, then the algorithm requires that the length of σ_i is at most equal to the number of rows $r_{ii} - r_{i-1,i}$ of $R_{i-1,i}$, and the diagram consists of σ_i in the top $r_{ii} - r_{i-1,i}$ rows and τ_{i-1} below this, possibly with some zero-length rows in between.

The proof of Theorem 4.1 is word for word identical to the proof given in [5], except that the lemmas 2, 3, and 4 of [5] are replaced with Theorem 7.3 from this paper, Corollary 6.5 of [3], and equation (2.5), respectively. The only point which requires a comment is that the modified proof will need that for certain proper birational maps $f : T \to S$ one has $f_*[\mathcal{O}_T] = [\mathcal{O}_S]$. This is true if T and S have rational singularities, which holds in all cases considered due to Lakshmibai and Magyar's result [16].

As mentioned above, the coefficients $c_\mu(r)$ depend only on the side lengths of the rectangles R_{ij}, not on the integers r_{ij} themselves. Given that the coefficients have this property, they are in fact uniquely given by the statement of Theorem 4.1 (see [5, §2.2]). Regarding the signs of the coefficients, we pose:

Conjecture 4.2. *The signs of the coefficients $c_\mu(r)$ alternate with the weight of $|\mu|$, i.e. $(-1)^{|\mu|-d(r)}\, c_\mu(r) \geq 0$.*

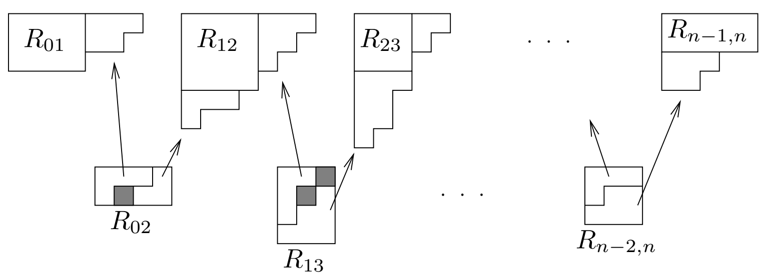
One particular case where this conjecture can be verified is when the rectangle diagram only has two non-empty rows, i.e. R_{ij} is empty when $j-i>2$. This case includes all varieties of complexes. When all rectangles below the second row are empty, the inductive element is given by $P_{\bar r} = G_{R_{02}} \otimes G_{R_{13}} \otimes \cdots \otimes G_{R_{n-2,n}}$.

Now for a rectangular partition R, the coproduct constants $d^R_{\sigma\tau}$ are given by the following simple rule. Define a *rook strip* to be a skew diagram which has at most one box in any row or column. Also, if τ is a partition which can be contained in R, let $\hat\tau$ denote τ rotated 180 degrees and placed in the bottom-right corner of R. We then have

$$d^R_{\sigma\tau} = \begin{cases} (-1)^{|\sigma|+|\tau|-|R|} & \text{if } \sigma \cup \hat\tau = R \text{ and } \sigma \cap \hat\tau \text{ is a rook strip;} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 4.3. *If R_{ij} is empty for $j-i>2$ then the formula is given by $P_r = \sum (-1)^{|\mu|-d(r)}\, G_{\mu_1} \otimes \cdots \otimes G_{\mu_n}$, where the sum is over all partitions $\mu_i = (R_{i-1,i} + \sigma_i, \tau_{i-1})$, such that $\sigma_i \cup \hat\tau_i = R_{i-1,i+1}$ and $\sigma_i \cap \hat\tau_i$ is a rook strip for all i.*

Notice that the relations among the side lengths of the rectangles imply that if $d^{R_{i-1,i+1}}_{\sigma_i,\tau_i} \neq 0$ then σ_i always fits on the right side of $R_{i-1,i}$ and τ_i fits below $R_{i,i+1}$, so the sequences of integers produced by the algorithm are always partitions.



In [5] it is conjectured that the coefficients $c_\mu(r)$ appearing in the cohomology formula (with $|\mu| = d(r)$) are given as the number of sequences of semistandard Young tableaux satisfying certain properties. It would be very interesting to generalize this conjecture to also give an expression for the more general coefficients defined in this paper.

5. Applications to Grothendieck polynomials

In this section we will sketch how to apply our formula to give new formulas for Grothendieck polynomials. Our development is analogous to [5, §2.3] and [4].

Let $E_\bullet$ be the sequence $F_1 \subset \cdots \subset F_n \to H_n \twoheadrightarrow \cdots \twoheadrightarrow H_1$ considered in Section 2, and let $w \in S_{n+1}$ be a permutation. Then $\Omega_w = \Omega_r(E_\bullet)$ where $r = (r_{ij})$ are the obvious rank conditions. Set $x_i = 1 - L_i^{-1}$ and $y_i = 1 - M_i$ where $L_i = \ker(H_i \to H_{i-1})$ and $M_i = F_i/F_{i-1}$. The double Grothendieck polynomial $\mathfrak{G}_w(x;y)$ then becomes a special case of the quiver formula:

$$\begin{aligned} \mathfrak{G}_w(x;y) &= [\mathcal{O}_{\Omega_r(E_\bullet)}] \\ &= \sum c_\mu(r)\, G_{\mu_1}(F_2 - F_1) \cdots G_{\mu_{n-1}}(F_n - F_{n-1}) \cdot G_{\mu_n}(H_n - F_n) \cdot \\ &\qquad G_{\mu_{n+1}}(H_{n-1} - H_n) \cdots G_{\mu_{2n-1}}(H_1 - H_2) \end{aligned}$$

Set $\tilde{x}_i = 1 - L_i = 1 - (1-x_i)^{-1} = -(\sum_{k\geq 1} x_i^k)$ and $\tilde{y}_i = 1 - M_i^{-1} = -(\sum_{k\geq 1} y_i^k)$. The formula can then be simplified using the identities

$$G_\lambda(F_i - F_{i-1}) = G_\lambda(M_i) = \begin{cases} (\tilde{y}_i)^a & \text{if } \lambda = (a) \text{ is a row with } a \text{ boxes} \\ 0 & \text{otherwise} \end{cases}$$

and

$$G_\lambda(H_{i-1} - H_i) = G_\lambda(-L_i) = \begin{cases} (\tilde{x}_i)^b & \text{if } \lambda = (1^b) \text{ is a column with } b \text{ boxes} \\ 0 & \text{otherwise.} \end{cases}$$

Using this we obtain a formula

$$\mathfrak{G}_w(x;y) = \sum c_w(a,b,\lambda)\, \tilde{y}_2^{a_2} \cdots \tilde{y}_n^{a_n}\, \tilde{x}_2^{b_2} \cdots \tilde{x}_n^{b_n}\, G_\lambda(x;y)\,. \tag{5.1}$$

The sum is over exponents $a_2, \dots, a_n$ and $b_2, \dots, b_n$, and a single partition λ, and $c_w(a,b,\lambda)$ is the coefficient $c_\mu(r)$ for the sequence of partitions

$$\mu = ((a_2), \dots, (a_n), \lambda, (1^{b_n}), \dots, (1^{b_2}))\,.$$

Notice that it is not clear from the expression (5.1) that $\mathfrak{G}_w(x;y)$ is a polynomial as opposed to a power series.

Now using the same arguments as in [4] we obtain

$$\mathfrak{G}_{1^m \times w}(x;y) = \sum c_w(a,b,\lambda)\, \tilde{y}_{2+m}^{a_2} \cdots \tilde{y}_{n+m}^{a_n}\, \tilde{x}_{2+m}^{b_2} \cdots \tilde{x}_{n+m}^{b_n}\, G_\lambda(x;y)$$

where $G_\lambda(x;y)$ is in variables $x_1, \dots, x_{n+m}$ and $y_1, \dots, y_{n+m}$. Letting m tend to infinity in this expression, it follows that

$$G_w(x;y) = \sum_\lambda c_w(0,0,\lambda)\, G_\lambda(x;y)\,.$$

Thus we see that when the stable Grothendieck polynomial G_w is expressed in the basis $\{G_\lambda\}$, the obtained coefficients are special cases of the quiver coefficients $c_\mu(r)$ defined in this paper. Lascoux has recently shown that $(-1)^{|\lambda|-\ell(w)}\, c_w(0,0,\lambda) \geq 0$ [18] which confirms a special case of Conjecture 4.2. In addition this identity shows that the structure constants of Γ are special cases of the quiver coefficients $c_\mu(r)$, cf. [3].

6. A generalized Jacobi-Trudi formula

Recall that when one expands the Jacobi-Trudi determinant for the Schur function s_λ after the first row, one gets $s_\lambda = \sum_{q\geq 0} (-1)^q\, s_{\lambda_1+q} \cdot s_{\mu/(1^q)}$ where $\mu = (\lambda_2, \lambda_3, \dots)$. In this section we will prove a generalization of this result for stable Grothendieck polynomials.

To state the formula in sufficient generality we need the following definition. If I is a sequence of integers and λ a partition, we write

$$G_{I/\!\!/\lambda} = \sum_{\nu,\mu} \delta_{I,\nu} d^\nu_{\lambda\mu}\, G_\mu \in \Gamma\,.$$

With this notation we have $\Delta G_I = \sum_\lambda G_{I/\!\!/\lambda} \otimes G_\lambda$. Notice that when $I = \nu$ is a partition, the element $G_{\nu/\!\!/\lambda}$ depends on both ν and λ and not just the skew diagram ν/λ between them. For example $G_{\lambda/\!\!/\lambda} = 1$ if and only if λ is the empty partition.

Theorem 6.1 (Jacobi-Trudi formula)**.** *If $a \in \mathbb{Z}$ is an integer and I is a sequence of integers, then*

$$G_{a,I} = G_a \cdot G_I + \sum_{q \geq 1,\, t \geq 0} (-1)^q \binom{q-1+t}{t} G_{a+q+t} \cdot G_{I /\!\!/ (1^q)}\,.$$

For proving this theorem, the following notation will get rid of a lot of special cases. We let $\left[{n \atop m} \right]$ be the usual binomial coefficient, except that we set $\left[{-1 \atop 0} \right] = 1$:

$$\left[{n \atop m} \right] = \begin{cases} \binom{n}{m} & \text{if } 0 \leq m \leq n, \\ 1 & \text{if } n = -1 \text{ and } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.1 then asserts that $G_{a,I} = \sum_{q,t \geq 0} (-1)^q \left[{q-1+t \atop t} \right] G_{a+q+t} \cdot G_{I /\!\!/ (1^q)}$, and we have $\left[{n \atop m} \right] = \left[{n-1 \atop m-1} \right] + \left[{n-1 \atop m} \right]$ whenever $m \leq n$.

Notice that since $G_{a,I} = \sum_\mu \delta_{I,\mu} G_{a,\mu}$ and $G_{I /\!\!/ (1^q)} = \sum_\mu \delta_{I,\mu} G_{\mu /\!\!/ (1^q)}$, it is enough to prove the theorem in case $I = \mu$ is a partition. We will give a bijective proof of the theorem when $a \geq \mu_1$. For this we need the following combinatorial objects. Recall that a skew diagram is called a *horizontal strip* if no two boxes are in the same column, and a *vertical strip* if no two boxes are in the same row. If both are true then the diagram is a rook strip.

Definition 6.2. A colored and marked Young diagram (CMYD) relative to a partition μ is a quadruple of partitions $D = (\lambda_0 \subset \lambda \subset \nu_0 \subset \nu)$ such that

(i) $\lambda \subset \mu$.
(ii) μ/λ_0 is a vertical strip.
(iii) ν/λ is a horizontal strip.
(iv) λ/λ_0 and ν/ν_0 are both rook strips.
(v) ν/ν_0 has no box in the top non-empty row of ν/λ.

We will regard a CMYD $D = (\lambda_0, \lambda, \nu_0, \nu)$ as the Young diagram for ν in which the boxes of λ are colored white and the boxes of ν/λ are gray; the boxes in λ/λ_0 and in ν/ν_0 are furthermore marked. The axioms (i)–(v) then say that all white boxes are contained in μ; the boxes in μ which are not white form a vertical strip; the gray boxes form a horizontal strip; the marked white boxes form a rook strip and the marked gray boxes form a rook strip; and finally the northernmost gray boxes are unmarked. Let

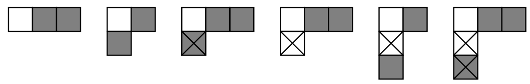
$$\begin{aligned} g(D) &= \#\text{ unmarked gray boxes in } D \ = |\nu_0/\lambda|\,, \\ w(D) &= \#\text{ unmarked white boxes in } D \ = |\lambda_0|\,, \\ u(D) &= \#\text{ unmarked boxes in } D \ = g(D) + w(D)\,, \text{ and} \\ m(D) &= \#\text{ marked boxes in } D \ = |\lambda/\lambda_0| + |\nu/\nu_0|\,. \end{aligned}$$

We will write $G_D = G_\nu$. By the coproduct Pieri rule of [3] or Theorem 2.2 we have $G_{\mu /\!\!/ (1^q)} = \sum (-1)^{m(D)} G_D$, the sum over all CMYDs relative to μ such that $w(D) = |\mu| - q$ and D has no gray boxes. Then using Lenart's Pieri rule [20, Thm. 3.2] or equation (2.2) we obtain

$$G_p \cdot G_{\mu /\!\!/ (1^q)} = \sum_D (-1)^{m(D)} G_D \tag{6.1}$$

where this sum is over all CMYDs D relative to μ with $g(D) = p$ and $w(D) = |\mu| - q$.

Example 6.3. If we take $\mu = (1,1)$, $p = 2$, and $q = 1$, we have the following 6 CMYDs:



It follows that $G_2 \cdot G_{1\,1/\!\!/1} = G_3 + G_{2\,1} - 2\,G_{3\,1} - G_{2\,1\,1} + G_{3\,1\,1}$.

Define the right vertical strip of μ to be the boxes in μ with no boxes to the right of them. We will say that a box in a CMYD D is in μ resp. in the right vertical strip of μ if this is true when the two diagrams are overlaid. We will be interested in the following four types of *special boxes* in D:

Type A: An unmarked gray box contained in μ which does not have a marked white box above it.

Type B: Any white box (marked or unmarked) contained in the right vertical strip of μ which has no box under it.

Type C: An unmarked gray box with a marked white box above it.

Type D: A marked gray box such that the box above it is in the right vertical strip of μ.

In Example 6.3 above, each diagram has exactly one special box. From left to right, these boxes have types B, A, D, B, C, and D. With this notion we can rewrite the right hand side of the formula of Theorem 6.1 as follows:

Lemma 6.4. *For any partition μ and integer $a \geq \mu_1$ we have*

$$\sum_{q,t \geq 0} (-1)^q \left[{q-1+t \atop t} \right] G_{a+q+t} \cdot G_{\mu/\!\!/(1^q)} = \sum_D (-1)^{|\mu|+w(D)+m(D)} \left[{g(D)-a-1 \atop u(D)-a-|\mu|} \right] G_D$$

where the sum is over all CMYDs relative to μ with no special boxes.

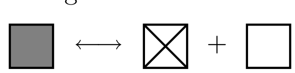
Proof. It follows from equation (6.1) that the asserted identity is true if we sum over all CMYDs relative to μ. We will prove that the terms for which D has special boxes cancel each other out in the right hand side. Notice that each column of a CMYD can have at most one special box. We will group each CMYD D for which the leftmost special box is of type A with two other CMYDs whose leftmost special boxes are of type B, such that the contributions from these three diagrams cancel. Similarly a diagram with a leftmost special box of type C will be grouped with two diagrams with leftmost special boxes of type D.

Notice that if D is a CMYD relative to μ such that $u(D) - a - |\mu| \geq 0$ and D contains a special box, then D has at least $a + 1$ gray boxes, so the top row of D contains an unmarked gray box which is outside μ. Notice also that since $\left[{n \atop m} \right] = \left[{n-1 \atop m-1} \right] + \left[{n-1 \atop m} \right]$ whenever $m \leq n$, we have

$$\left[{g(D)-a-1 \atop u(D)-a-|\mu|} \right] = \left[{g(D)-a-2 \atop u(D)-a-|\mu|-1} \right] + \left[{g(D)-a-2 \atop u(D)-a-|\mu|} \right] \tag{6.2}$$

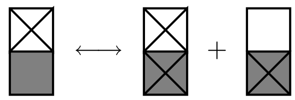
for any diagram D such that $w(D) < |\mu|$.

Now let D be a CMYD whose leftmost special box is of type A. The conditions for a type A box then make it possible to change this box into a white box or a marked white box, while the diagram continues to be a CMYD. Here it is important that the top row of D contains at least one unmarked gray boxes outside μ, since this ensures that the modified diagram satisfies axiom (v).



The signs of the contributions from the two new diagrams are the opposite of the sign of the contribution from D. Since $w(D) < |\mu|$, the contributions from all three diagrams therefore add to zero by equation (6.2). Notice that any special box of type B can be changed to a gray box. Therefore all diagrams with a leftmost special box of type B get canceled in this way.

Now suppose the leftmost special box in D is of type C. In this case the box can be changed to a marked gray box while the box above is either marked or unmarked white.



Again the new diagrams give contributions of opposite sign from that of D, and $w(D) < |\mu|$, so the contributions of all three diagrams cancel by equation (6.2). Finally all diagrams with a leftmost special box of type D are taken care of in this way, since any diagram with a type D box can be changed so the special box turns into type C. □

Lemma 6.5. *Let D be a CMYD with no special boxes and assume $u(D) \geq \mu_1 + |\mu|$. Then $D = (\mu, \mu, \nu, \nu)$ where $\nu = (g(D), \mu) = (g(D), \mu_1, \mu_2, \dots)$.*

Proof. We start by observing that D has no marked white boxes. If D has such a box, then since it is not special, there must be a gray box below it. But this gray box must then be special of type C or D, a contradiction. Notice also that no unmarked gray boxes can be contained in μ, since these would necessarily be special of type A.

Now suppose D contains a marked gray box, and consider the northernmost such box. Since this box is not special (and not in the top row of D), the white box above it is not in the right vertical strip of μ. Now consider the row of boxes in D to the right of this white box. If this row contains a box in the right vertical strip of μ, then this would necessarily be a special box of type B. We conclude that if D contains a marked gray box then some box northeast of this box is contained in μ but not in D.

Now assume that μ is not contained in D and consider the northernmost row where D is missing boxes from μ. Since D contains at least μ_1 gray boxes, this can't be the top row, and the row above must contain a box in the right vertical strip of μ which has no box below it. Since this box can't be marked gray by the argument above, it must be special of type B, again a contradiction.

We conclude that μ is contained in D and that all boxes from μ are white. To prevent these white boxes from being special, there must furthermore be a gray box in each column of D. This proves the result. □

The preceding two lemmas essentially prove Theorem 6.1 when $I = \mu$ is a partition and $a \geq \mu_1$. For the general case of the theorem we will also need the following lemma. Let $h_i(x)$ denote the complete symmetric function of degree i.

Lemma 6.6. *For any integer $k \in \mathbb{Z}$ we have $G_k(x) = (1 - G_1(x)) \cdot \sum_{i \geq 0} h_{k+i}(x)$.*

Proof. If $k \geq 1$, it follows from [20, Thm. 2.2] that $G_k(x) = \sum_{p \geq 0} (-1)^p \, s_{(k,1^p)}(x)$. Alternatively this can be deduced from equation (2.4), see e.g. [3, §6]. Notice in particular that $1 - G_1(x) = \sum_{p \geq 0} (-1)^p \, e_p(x)$. For $k \geq 1$ the lemma therefore

follows from the identity $\sum_{i=0}^{p}(-1)^i\, h_{k+i}\, e_{p-i} = s_{(k,1^p)}$. When $k \leq 0$ the lemma is true because $\sum_{p\geq 0}(-1)^p\, e_p$ is the inverse power series to $\sum_{i\geq 0} h_i$. □

Proof of Theorem 6.1. Suppose at first that $a \geq \mu_1$. If D is a CMYD relative to μ with no special boxes such that its coefficient $\left[\begin{smallmatrix} g(D)-a-1 \\ u(D)-a-|\mu| \end{smallmatrix}\right]$ is non-zero, then since $u(D) \geq a+|\mu|$ we conclude by Lemma 6.5 that $D = (\mu,\mu,\nu,\nu)$ where $\nu = (g(D),\mu)$. But then we have $w(D) = |\mu|$ and $\left[\begin{smallmatrix} g(D)-a-1 \\ g(D)-a \end{smallmatrix}\right] \neq 0$, so $g(D) = a$. The theorem therefore follows from Lemma 6.4 in all cases where $a \geq \mu_1$.

For the general case it is enough to show that

$$\mathcal{G}_{a,\mu}(x_1,\dots,x_n) = \sum_{q,t\geq 0} (-1)^q \left[\begin{smallmatrix} q-1+t \\ t \end{smallmatrix}\right] G_{\mu/\!\!/(1^q)}(x_1,\dots,x_n)\cdot G_{a+q+t}(x_1,\dots,x_n)$$

where $n \geq 1+\max(\ell(\mu),-a)$; this is sufficient because any partition λ such that G_λ occurs in either side of the claimed identity must have length at most $\ell(\mu)+1$, and the stable Grothendieck polynomials for partitions of such lengths are linearly independent when applied to n variables. For the rest of this proof we will let x denote the n variables $x_1,\dots,x_n$.

Let $\mathcal{G}_\mu^{(i)}$ be the cofactor obtained by removing the first row and the $i+1$'st column of the determinant defining $\mathcal{G}_{a,\mu}(x)$. Notice that this does not depend on a, and we have

$$\mathcal{G}_{a,\mu}(x) = \sum_{i=0}^{n-1}(-1)^i\,\mathcal{G}_\mu^{(i)}(x)\cdot h_{a+i}(x)\,. \tag{6.3}$$

Now using Lemma 6.6 we obtain

$$\begin{aligned}
&\sum_{q,t\geq 0}(-1)^q \left[\begin{smallmatrix} q-1+t \\ t \end{smallmatrix}\right] G_{\mu/\!\!/(1^q)}(x)\cdot G_{a+q+t}(x) \\
&= \sum_{q,t\geq 0,\, i\geq q+t}(-1)^q \left[\begin{smallmatrix} q-1+t \\ t \end{smallmatrix}\right] G_{\mu/\!\!/(1^q)}(x)\cdot(1-G_1(x))\cdot h_{a+i}(x) \\
&= \sum_{i\geq 0}\left((1-G_1(x))\cdot\sum_{q+t\leq i}(-1)^q \left[\begin{smallmatrix} q-1+t \\ t \end{smallmatrix}\right] G_{\mu/\!\!/(1^q)}(x)\right)\cdot h_{a+i}(x)\,.
\end{aligned} \tag{6.4}$$

Since (6.3) is equal to (6.4) for all $a \geq \mu_1$, the theorem follows from the following lemma. □

Lemma 6.7. *Let $f_j \in \mathbb{Z}[[x_1,\dots,x_n]]$ be a power series for each $j \geq 0$ and assume that*

$$\sum_{j\geq 0} h_{a+j}(x_1,\dots,x_n)\cdot f_j = 0 \tag{6.5}$$

holds for all sufficiently large $a \in \mathbb{N}$. Then (6.5) is true for all $a \geq 1-n$.

Proof. Since the form of each fixed degree in (6.5) must be zero, we can assume that each f_j is a polynomial and that $f_j = 0$ for $j > d$ for some $d \in \mathbb{N}$. Assume at first that $d < n$ and let (6.5) be true whenever $a \geq N$. By assumption we then

have

$$\begin{bmatrix} h_{N+d} & h_{N+d+1} & \dots & h_{N+2d} \\ h_{N+d-1} & h_{N+d} & \dots & h_{N+2d-1} \\ \vdots & \vdots & & \vdots \\ h_N & h_{N+1} & \dots & h_{N+d} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

Since the determinant of the matrix is the Schur polynomial $s_{(N+d)^{d+1}}(x_1,\dots,x_n) \neq 0$, we conclude that each $f_j = 0$.

Now assume $d \geq n$. If $a \geq 1-n$ then since $a+d \geq 1$ we get

$$h_{a+d}(x_1,\dots,x_n) = \sum_{j=0}^{d-1} (-1)^{d-j+1} h_{a+j}(x_1,\dots,x_n) e_{d-j}(x_1,\dots,x_n) .$$

So if we put $g_j = f_j + (-1)^{d-j+1} e_{d-j}(x_1,\dots,x_n) f_d$, the left hand side of (6.5) is equal to

$$\sum_{j=0}^{d-1} h_{a+j}(x_1,\dots,x_n) \cdot g_j .$$

Since this is equal to zero for all large a, we conclude it is zero for all $a \geq 1-n$ by induction on d. $\square$

7. A Gysin formula

In this section we prove of a K-theory parallel of a Gysin formula of Pragacz [21, 13]. We start with a Lemma which indicates that the classes $G_k(F)$ are the right K-theoretic generalizations of *Segre classes* of a vector bundle F.

Lemma 7.1. *Let F be a vector bundle of rank f over a variety X. Let $\pi : \mathbb{P}^*(F) \to X$ be the dual projective bundle of F and let Q be the tautological quotient of $\pi^* F$. Then for any $k \in \mathbb{Z}$ we have $\pi_*(G_k(Q)) = G_{k-f+1}(F)$.*

Proof. This is clearly true if $k \leq 0$. For $1 \leq k \leq f-1$ we write $G_k(Q) = (1-Q^{-1})^k = 1 + \sum_{i=1}^k (-1)^i Q^{-i}$, and the lemma follows because $R^j \pi_*(Q^{-i}) = 0$ for all j and $1 \leq i \leq f-1$.

Finally, if $k \geq f$ we set $E = \mathcal{O}_X^{\oplus k}$ and form the bundle $H = \mathrm{Hom}(E,F) \to X$. Then construct the fiber square:

$$\begin{tikzcd} Y \arrow[r, "\pi'"] \arrow[d] & H \arrow[d] \\ \mathbb{P}^*(F) \arrow[r, "\pi"] & X \end{tikzcd}$$

We will suppress pullback notation for vector bundles. By [5, Lemma 1] the locus $Z(E \to Q)$ in Y is mapped birationally onto $\Omega_{f-1}(E \to F) \subset H$. Using Theorem 2.3 and the fact that determinantal varieties have rational singularities [14, 16] we therefore get

$$\pi'_*(G_k(Q)) = \pi'_*([\mathcal{O}_{Z(E\to Q)}]) = [\mathcal{O}_{\Omega_{f-1}(E\to F)}] = G_{k-f+1}(F) .$$

Since pullback along the vertical maps are isomorphisms which are compatible with the horizontal pushforward maps, the lemma follows from this. $\square$

In the above lemma we have applied the usual proof of the Thom-Porteous formula in the opposite way. Alternatively one can prove the identity $G_p(x_1,\dots,x_n) = 1-\prod_{i=1}^n(1-x_i)\cdot\sum_{j\geq 0}\binom{n+p-1}{n+j}h_j(x_1-1,\dots,x_n-1)$ by induction on n. The lemma follows from this using the identities $\pi_*(Q^i) = S^iF$, $R^j\pi_*(Q^i) = 0$ for $0 < j < f-1$, and $R^{f-1}\pi_*(Q^{-f-i}) = \det(F^\vee)\otimes S^iF^\vee$.

Lemma 7.2. *Let F and E be vector bundles. For any integers m, i with $i \geq \operatorname{rank}(E)$ we have $\sum_{k\geq 0} G_{m+k}(F)\,G_{i/\!\!/k}(-E) = G_{m+i}(F-E)$.*

Proof. Notice that for any $p\in\mathbb{Z}$ we have $G_{p/\!\!/0} = G_p$ and $G_{p/\!\!/k} = G_{p-k} - G_{p-k+1}$ for all $k > 0$. Since $G_j(-E) = 0$ for $j > \operatorname{rank}(E)$ this means that $G_{p/\!\!/k}(-E) = G_{p-k}(-E) - G_{p-k+1}(-E)$ if either $k > 0$, or $k \geq 0$ and $p \geq \operatorname{rank}(E)$.

If $m \geq 0$ then $G_{m+i/\!\!/k}(-E) = 0$ for $k < m$ and $G_{m+i/\!\!/k}(-E) = G_{i/\!\!/k-m}(-E)$ for $k \geq m$, so $G_{m+i}(F-E) = \sum_{k\geq 0} G_k(F)\,G_{m+i/\!\!/k}(-E) = \sum_{k\geq 0} G_{m+k}(F)\,G_{i/\!\!/k}(-E)$ as required.

For $m \leq 0$ we have $\sum_{k=0}^{-m} G_{m+k}(F)\,G_{i/\!\!/k}(-E) = \sum_{k=0}^{-m} G_{i/\!\!/k}(-E) = G_{m+i}(-E)$, so the left hand side in the lemma is $G_{m+i}(-E) + \sum_{k\geq 1-m} G_{m+k}(F)\,G_{i/\!\!/k}(-E) = G_{m+i}(-E) + \sum_{k\geq 1} G_k(F)\,G_{m+i/\!\!/k}(-E) = G_{m+i}(F-E)$ as required. $\square$

Theorem 7.3. *Let E and F be bundles on X of ranks e and f. Let $f = d+q$ and let $\pi : \operatorname{Gr}(d,F)\to X$ be the Grassmann bundle of d-planes in F with universal exact sequence $0\to A\to\pi^*F\to Q\to 0$. Let $I = (I_1,\dots,I_q)$ and $J = (J_1, J_2,\dots)$ be sequences of integers such that $I_j \geq e$ for all j. Then*

$$\pi_*(G_I(Q-E)\cdot G_J(A-E)) = G_{I-(d)^q,J}(F-E)\,.$$

Proof. Let $\mathrm{F}\ell(d,f-1;F)$ be the variety of partial flags $A\subset H\subset F$ such that A has rank d and H has rank $f-1$. Then form the commutative diagram from [15]:

$$\begin{tikzcd} \mathrm{F}\ell(d,f-1;F) \arrow[r] \arrow[d] & \operatorname{Gr}(d,F) \arrow[d, "\pi"] \\ \mathbb{P}^*(F) \arrow[r] & X \end{tikzcd}$$

The formula can now be proved by calculating the pushforward to X of the class $G_{I_1+q-1}(F/H-E)\cdot G_{\tilde I}(H/A-E)\cdot G_J(A-E)$ in two different ways, using descending induction on d. Here $\tilde I$ is the sequence $(I_2,\dots,I_q)$.

We are therefore reduced to the case $d = f-1$ where $\operatorname{Gr}(d,F) = \mathbb{P}^*(F)$ is a projective bundle. Notice that since Q is now a line bundle we have $G_k(Q) = (1-[Q^\vee])^k$ for $k\geq 0$. Using this we get the following identities in $K^\circ\mathbb{P}^*(F)$.

$$\begin{aligned}
&G_i(Q-E)\cdot G_J(A-E) = G_i(Q-E)\cdot G_J(F-E\oplus Q)\\
&\quad= \sum_{k\geq 0} G_k(Q)\cdot G_{i/\!\!/k}(-E)\cdot \sum_{\ell\geq 0} G_{(1^\ell)}(-Q)\cdot G_{J/\!\!/(1^\ell)}(F-E)\\
&\quad= \sum_{k,\ell\geq 0} G_k(Q)\cdot G_\ell(Q^\vee)\cdot G_{i/\!\!/k}(-E)\cdot G_{J/\!\!/(1^\ell)}(F-E)\\
&\quad= \sum_{k,\ell\geq 0} (-1)^\ell [Q]^\ell G_{k+\ell}(Q)\cdot G_{i/\!\!/k}(-E)\cdot G_{J/\!\!/(1^\ell)}(F-E)\\
&\quad= \sum_{k,\ell\geq 0} (-1)^\ell (1-G_1(Q))^{-\ell}\cdot G_{k+\ell}(Q)\cdot G_{i/\!\!/k}(-E)\cdot G_{J/\!\!/(1^\ell)}(F-E)\\
&\quad= \sum_{k,\ell\geq 0} (-1)^\ell \sum_{t\geq 0} \left[\begin{smallmatrix}\ell-1+t\\ t\end{smallmatrix}\right] G_1(Q)^t\cdot G_{k+\ell}(Q)\cdot G_{i/\!\!/k}(-E)\cdot G_{J/\!\!/(1^\ell)}(F-E)\\
&\quad= \sum_{k,\ell,t\geq 0} (-1)^\ell \left[\begin{smallmatrix}\ell-1-t\\ t\end{smallmatrix}\right] G_{k+\ell+t}(Q)\cdot G_{i/\!\!/k}(-E)\cdot G_{J/\!\!/(1^\ell)}(F-E)
\end{aligned}$$

The step replacing $(1-G_1(Q))^{-\ell}$ with its power series expansion is valid since $G_1(Q)^t$ is zero for $t > \dim \mathbb{P}^*(F)$. Now using Lemma 7.1, Lemma 7.2, and Theorem 6.1 we get

$$\begin{aligned}
&\pi_*(G_i(Q-E)\cdot G_J(A-E))\\
&\quad= \sum_{k,\ell,t\geq 0} (-1)^\ell \left[\begin{smallmatrix}\ell-1+t\\ t\end{smallmatrix}\right] G_{k+\ell+t-d}(F)\cdot G_{i/\!\!/k}(-E)\cdot G_{J/\!\!/(1^\ell)}(F-E)\\
&\quad= \sum_{\ell,t\geq 0} (-1)^\ell \left[\begin{smallmatrix}\ell-1+t\\ t\end{smallmatrix}\right] G_{\ell+t+i-d}(F-E)\cdot G_{J/\!\!/(1^\ell)}(F-E)\\
&\quad= G_{i-d,J}(F-E)
\end{aligned}$$

which is what we want to prove. ∎

Continuing the remark after Lemma 7.1, notice that Theorem 2.3 is a consequence of Theorem 7.3 once we prove that the structure sheaf of a zero section $Z(E\to F)$ is given by $G_{(e)^f}(F-E)$, see e.g. [10, §14.4]. This in turn follows from [3, eqn. (7.1)].

We will finish this paper with the following somewhat surprising consequence of Theorem 7.3.

Corollary 7.4. *Let $\pi : \mathrm{Gr}(d,F)\to X$ be a Grassmann bundle with tautological quotient bundle Q of rank q. For partitions λ and μ of lengths at most q, we have*

$$\pi_*(G_\lambda(Q))\cdot\pi_*(G_\mu(Q)) = \pi_*(G_\lambda(Q)\cdot G_\mu(Q)) + \sum_{\ell(\nu)>q} c^\nu_{\lambda\mu}\, G_{\tilde\nu}(F)$$

where the sum is over all partitions ν of length strictly greater than q and $\tilde\nu$ denotes ν with the first d columns removed. In particular, if $\ell(\lambda)+\ell(\mu)\leq q$ then we get $\pi_*(G_\lambda(Q)\cdot G_\mu(Q)) = \pi_*(G_\lambda(Q))\cdot\pi_*(G_\mu(Q))$.

In other words, π_* behaves like a ring homomorphism for short partitions!

Proof. In [3, §7] it is shown that the linear map $\Gamma\to\Gamma$ defined by $G_\nu\mapsto G_{\tilde\nu}$ is a ring homomorphism. Using this we get $\pi_*(G_\lambda(Q))\cdot\pi_*(G_\mu(Q)) = G_{\tilde\lambda}(F)\cdot G_{\tilde\mu}(F) = \sum_\nu c^\nu_{\lambda\mu} G_{\tilde\nu}(F)$. On the other hand we have $G_\lambda(Q)\cdot G_\mu(Q) = \sum_{\ell(\nu)\leq q} c^\nu_{\lambda\mu} G_\nu(Q)$, so $\pi_*(G_\lambda(Q)\cdot G_\mu(Q)) = \sum_{\ell(\nu)\leq q} c^\nu_{\lambda\mu} G_{\tilde\nu}(F)$. The corollary follows from this. ∎

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